Fedosov's formal symplectic groupoids and contravariant connections

Alexander V. Karabegov*

Department of Mathematics and Computer Science, Abilene Christian University, ACU Box 28012, 252 Foster Science Building, Abilene, TX 79699-8012, United States

Abstract

Using Fedosov's approach we give a geometric construction of a formal symplectic groupoid over any Poisson manifold endowed with a torsion-free Poisson contravariant connection. In the case of Kähler–Poisson manifolds this construction provides, in particular, the formal symplectic groupoids with separation of variables. We show that the dual of a semisimple Lie algebra does not admit torsion-free Poisson contravariant connections.
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1. Introduction

A symplectic groupoid over a Poisson manifold M is a symplectic manifold Σ endowed with a partially defined multiplication and the source, target, inverse, and unit mappings satisfying several axioms. In particular, the source and the target mappings are a Poisson and an anti-Poisson mapping from Σ to M, respectively. Symplectic groupoids play the rôle of semiclassical counterparts of associative algebras treated as quantum objects. Symplectic groupoids were introduced independently by Karasëv [11], Weinstein [14,3], and Zakrzewski [16]. There is a corresponding notion of a formal symplectic groupoid on the formal neighborhood (Σ, Λ) of a

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*Tel.: +1 325 672 3309; fax: +1 325 674 6753.
E-mail address: axk02d@acu.edu.

Lagrangian submanifold Λ of a symplectic manifold Σ whose principal example is the formal neighborhood (Σ, Λ) of the Lagrangian unit space Λ of a symplectic groupoid on a symplectic manifold Σ (see [10]). Formal symplectic groupoids were first introduced in [2] in terms of formal generating functions of their (formal) Lagrangian product spaces. It was shown in [10] that to each natural deformation quantization on a Poisson manifold M there corresponds a canonical formal symplectic groupoid on (T^*M, Z), where Z is the zero section of T^*M. The main result of [2] is the description of the formal symplectic groupoid of Kontsevich deformation quantization. The formal symplectic groupoid of Fedosov's star-product was described in [9]. This paper is motivated by the following observation. On the one hand, it is known that deformation quantizations with separation of variables (also known as deformation quantizations of the Wick type, see [7] and [1]) are a particular case of Fedosov's deformation quantizations (see [12]). On the other hand, it was shown in [10] that the corresponding formal symplectic groupoids "with separation of variables" can be naturally extended from Kähler manifolds to Kähler–Poisson manifolds, while it is impossible to extend the star-products with separation of variables to the Kähler–Poisson manifolds in a naive direct way (see [8]). In this paper we show that the construction of the formal symplectic groupoids of Fedosov's deformation quantizations from [9] can be naturally extended to the Poisson manifolds endowed with a torsion-free Poisson contravariant connection. We call the formal symplectic groupoids obtained via this construction Fedosov's formal symplectic groupoids.

On a Kähler–Poisson manifold, there is a natural torsion-free Poisson contravariant connection which we call the Kähler–Poisson contravariant connection. We show that Fedosov's formal symplectic groupoid constructed with the use of the Kähler–Poisson contravariant connection is a formal symplectic groupoid with separation of variables.

Any symplectic manifold admits symplectic (torsion-free) connections and therefore Poisson torsion-free contravariant connections. However, this is not the case for general Poisson manifolds. We prove that the dual space of a semisimple Lie algebra does not admit a torsion-free Poisson contravariant connection.

2. Linear contravariant connections

Contravariant derivatives were introduced by Vaisman in [13]. The corresponding notion of a contravariant connection was extensively studied by Fernandes in [5].

Let M be a Poisson manifold endowed with the Poisson bivector field Π. Then the Poisson bracket of functions $f, g \in C^\infty(M)$ is given by

$$\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g).$$

Define a bundle map $\# : T^*M \to TM$ by the formula

$$\langle \beta, \#\alpha \rangle = \Pi(\alpha, \beta),$$

where $\alpha, \beta \in \Omega^1(M)$ are 1-forms on M and $\langle \cdot, \cdot \rangle$ is the natural pairing of T^*M and TM. It is known that on the space $\Omega^1(M)$ of 1-forms on M there is a Lie bracket

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - \mathrm{d}\Pi(\alpha, \beta),$$

where $\alpha, \beta \in \Omega^1(M)$ and $\mathcal{L}$ denotes the Lie derivative. If $\alpha = \mathrm{d}f$ and $\beta = \mathrm{d}g$ for $f, g \in C^\infty(M)$, then

$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\}. \tag{1}$$

The mapping # induces a Lie algebra homomorphism from $\Omega^1(M)$ to the Lie algebra of vector fields on M, Vect(M), so that for $\alpha, \beta \in \Omega^1(M)$

$$\#[\alpha, \beta] = [\#\alpha, \#\beta]. \tag{2}$$

A contravariant connection $\nabla^\bullet$ on a vector bundle E on M is a bilinear mapping $\nabla^\bullet : \Gamma(T^*M) \otimes \Gamma(E) \to \Gamma(E)$ satisfying the axioms

(i) $\nabla^{f\alpha} s = f\nabla^\alpha s$;
(ii) $\nabla^\alpha(fs) = f\nabla^\alpha s + \#\alpha(f)s$,

where $f \in C^\infty(M), \alpha \in \Omega^1(M)$, and $s \in \Gamma(E)$.

A covariant connection $\nabla_\bullet$ on E induces a contravariant connection $\nabla^\bullet$ as follows:

$$\nabla^\alpha = \nabla_{\#\alpha}. \tag{3}$$

It is known that there exist contravariant connections that are not induced by a covariant connection.

Let $\nabla^\bullet$ be a contravariant connection on T^*M. Its torsion T^∇ and curvature R^∇ are defined by the formulas

$$T^\nabla(\alpha, \beta) = \nabla^\alpha\beta - \nabla^\beta\alpha - [\alpha, \beta]$$

and

$$R^\nabla(\alpha, \beta)\gamma = \nabla^\alpha\nabla^\beta\gamma - \nabla^\beta\nabla^\alpha\gamma - \nabla^{[\alpha,\beta]}\gamma,$$

where α, β, γ are 1-forms on M. The transposed contravariant connection ${}^t\nabla^\bullet$ of a contravariant connection $\nabla^\bullet$ is defined as follows:

$${}^t\nabla^\alpha \beta = \nabla^\beta\alpha + [\alpha, \beta],$$

so that $T^\nabla(\alpha, \beta) = \nabla^\alpha\beta - {}^t\nabla^\alpha \beta$. The transposed connection of ${}^t\nabla^\bullet$ is $\nabla^\bullet$.

To any linear covariant connection $\nabla_\bullet$ on TM there corresponds a covariant connection on T^*M which will be denoted also by $\nabla_\bullet$. In local coordinates $\{x^i\}$

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k \qquad \text{and} \qquad \nabla_{\partial_i}\mathrm{d}x^j = -\Gamma_{ik}^j\mathrm{d}x^k,$$

where $\partial_i = \partial/\partial x^i$ and Γ_{ij}^k are the Christoffel symbols of $\nabla_\bullet$. Similarly, to any contravariant connection $\nabla^\bullet$ on TM there corresponds a contravariant connection on T^*M which will be denoted also by $\nabla^\bullet$. In local coordinates

$$\nabla^{\mathrm{d}x^i}\partial_j = \Gamma_j^{ik}\partial_k \qquad \text{and} \qquad \nabla^{\mathrm{d}x^i}\mathrm{d}x^j = -\Gamma_k^{ij}\mathrm{d}x^k,$$

where Γ_k^{ij} are the Christoffel symbols of $\nabla^\bullet$. In particular, if a contravariant connection $\nabla^\bullet$ is induced by a covariant connection $\nabla_\bullet$ according to (3), then

$$\Gamma_k^{ij} = \pi^{il}\Gamma_{lk}^j,$$

where π^{ij} is the Poisson tensor corresponding to Π.

To a given covariant connection $\nabla_\bullet$ on TM there corresponds the transposed connection ${}^t\nabla_\bullet$ on TM such that ${}^t\Gamma_{ij}^k = \Gamma_{ji}^k$, where Γ_{ij}^k and ${}^t\Gamma_{ij}^k$ are the Christoffel symbols of $\nabla_\bullet$ and ${}^t\nabla_\bullet$, respectively. For vector fields $X, Y \in$ Vect(M)

$$ {}^t\nabla_X Y = \nabla_Y X + [X, Y], \tag{4}$$

so that the torsion $T_\nabla(X, Y)$ of $\nabla_\bullet$ is given by the formula

$$T_\nabla(X, Y) = \nabla_X Y - {}^t\nabla_X Y.$$

The transposed connection of ${}^t\nabla_\bullet$ is $\nabla_\bullet$. Given a covariant connection $\nabla_\bullet$, one can construct a pair of contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ induced by $\nabla_\bullet$ and ${}^t\nabla_\bullet$, respectively, according to (3). The connection ${}^\dagger\nabla^\bullet$ is in general different from the transposed contravariant connection ${}^t\nabla^\bullet$. For 1-forms $\alpha, \beta \in \Omega^1(M)$ it follows from (2)–(4) that

$$ {}^\dagger\nabla^\alpha \, \#\beta - \nabla^\beta \#\alpha - \#[\alpha, \beta] = 0. \tag{5}$$

A pair $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ of contravariant connections satisfying (5) is a natural counterpart of a single covariant connection. We will call such connections *associated*. In general, one cannot recover one of the associated connections $\nabla^\bullet, {}^\dagger\nabla^\bullet$ from the other.

Lemma 1. *Contravariant connections $\nabla^\bullet, {}^\dagger\nabla^\bullet$ are associated if and only if in local coordinates their Christoffel symbols Γ_k^{ij} and ${}^\dagger\Gamma_k^{ij}$, respectively, satisfy the following condition:*

$$\pi^{ik} \Gamma_k^{jl} = \pi^{jk} \, {}^\dagger\Gamma_k^{il}. \tag{6}$$

Proof. It is enough to check (5) for $\alpha = dx^i$ and $\beta = dx^j$. Using (1) and the Jacobi identity for the Poisson tensor π^{ij} we obtain that

$$ {}^\dagger\nabla^{dx^i} \#dx^j - \nabla^{dx^j} \#dx^i - \#[dx^i, dx^j] = {}^\dagger\nabla^{dx^i}(\pi^{jk}\partial_k) $$

$$ - \nabla^{dx^j}(\pi^{ik}\partial_k) - \frac{\partial \pi^{ij}}{\partial x^s} \pi^{sk}\partial_k = \pi^{is} \frac{\partial \pi^{jk}}{\partial x^s} \partial_k + \pi^{jk} \, {}^\dagger\Gamma_k^{il}\partial_l $$

$$ - \pi^{js} \frac{\partial \pi^{ik}}{\partial x^s} \partial_k - \pi^{ik}\Gamma_k^{jl}\partial_l - \frac{\partial \pi^{ij}}{\partial x^s}\pi^{sk}\partial_k = \left(\pi^{jk} \, {}^\dagger\Gamma_k^{il} - \pi^{ik}\Gamma_k^{jl} \right)\partial_l, $$

whence the lemma follows. $\square$

Lemma 2. *A covariant connection $\nabla_\bullet$ respects the Poisson bivector field Π if and only if the contravariant connection ${}^\dagger\nabla^\bullet$ induced by ${}^t\nabla_\bullet$ is torsion-free.*

Proof. Denote by $T^{\dagger\nabla}$ the torsion of the connection ${}^\dagger\nabla^\bullet$. Taking into account that ${}^\dagger\Gamma_k^{ij} = \pi^{is} \, {}^t\Gamma_{sk}^j = \pi^{is}\Gamma_{ks}^j$ we get

$$ T^{\dagger\nabla}(dx^i, dx^j) = \left(-{}^\dagger\Gamma_k^{ij} + {}^\dagger\Gamma_k^{ji} - \frac{\partial \pi^{ij}}{\partial x^k} \right) dx^k $$

$$ = -\left(\frac{\partial \pi^{ij}}{\partial x^k} + \Gamma_{ks}^i \pi^{sj} + \Gamma_{ks}^j \pi^{is} \right) dx^k = -\left(\nabla_{\partial_k} \pi^{ij} \right) dx^k, $$

whence the claim follows. $\square$

Thus a pair of associated contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ such that ${}^\dagger\nabla^\bullet$ is torsion-free can be thought of as a natural counterpart of a covariant connection $\nabla_\bullet$ which respects the Poisson bivector field Π.

Lemma 3. *If contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ are associated then for $\alpha, \beta, \gamma \in \Omega^1$*

$$\left(\nabla^\alpha \Pi \right)(\beta, \gamma) = -\Pi\left(\alpha, T^{\dagger\nabla}(\beta, \gamma) \right). \tag{7}$$

In particular, if $^\dagger\nabla^\bullet$ is torsion-free then the contravariant connection $\nabla^\bullet$ respects the Poisson bivector field Π.

Proof. Using (6) we obtain (7) by the following calculation:

$$\nabla^{\mathrm{d}x^m}\pi^{ij} = \pi^{mk}\frac{\partial\pi^{ij}}{\partial x^k} + \Gamma_k^{mi}\pi^{kj} + \Gamma_k^{mj}\pi^{ik}$$
$$= -\pi^{mk}\left(-{}^\dagger\Gamma_k^{ij} + {}^\dagger\Gamma_k^{ji} - \frac{\partial\pi^{ij}}{\partial x^k}\right). \quad \square$$

A contravariant connection $\nabla^\bullet$ which respects the Poisson bivector field Π is called Poisson in [5]. A connection $\nabla^\bullet$ is Poisson if and only if it satisfies the condition

$$\#\nabla^\alpha\beta = \nabla^\alpha\#\beta \tag{8}$$

for any $\alpha, \beta \in \Omega^1(M)$. It was shown in [5] that on any Poisson manifold there exist Poisson contravariant connections.

Assume once again that a pair $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ of contravariant connections is induced by a pair $(\nabla_\bullet, {}^t\nabla_\bullet)$ of covariant connections. If $\nabla_\bullet$ is torsion-free, then $\nabla^\bullet = {}^\dagger\nabla^\bullet$. Thus a contravariant connection $\nabla^\bullet$ which satisfies the condition

$$\nabla^\alpha\#\beta - \nabla^\beta\#\alpha - \#[\alpha, \beta] = 0 \tag{9}$$

can be thought of as a natural counterpart of a torsion-free covariant connection.

If a contravariant connection $\nabla^\bullet$ satisfies condition (9) and is torsion-free, it can be considered as an analogue of a torsion-free covariant connection which respects Π.

Lemma 4. *A torsion-free contravariant connection $\nabla^\bullet$ respects the Poisson bivector field Π if and only if it satisfies (9).*

Proof. Assume that $\nabla^\bullet$ respects Π and therefore satisfies (8). Then (9) immediately follows from (8) applied to the condition that $\nabla^\bullet$ is torsion-free. Conversely, if $\nabla^\bullet$ is torsion-free and satisfies (9) it follows from Lemma 3 that $\nabla^\bullet$ respects Π.

It follows from Lemma 4 that if $\nabla^\bullet$ is a torsion-free Poisson contravariant connection on M, then it is associated to itself and one can consider the pair $(\nabla^\bullet, \nabla^\bullet)$ as an example of a pair of associated connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ such that $^\dagger\nabla^\bullet$ is torsion-free. Conversely, if contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ on M are associated and $^\dagger\nabla^\bullet$ is torsion-free then there exists a torsion-free Poisson connection on M.

Proposition 1. *If contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ are associated and $^\dagger\nabla^\bullet$ is torsion-free then*

$$\hat{\nabla}^\bullet = \frac{1}{3}\left(\nabla^\bullet + {}^t\nabla^\bullet + {}^\dagger\nabla^\bullet\right)$$

is a torsion-free Poisson contravariant connection.

Proof. It is clear that $\hat{\nabla}^\bullet$ is torsion-free. Taking into account Lemmas 1, 3 and 4 we see that it remains to show that the Christoffel symbols of the connection $\hat{\nabla}^\bullet$,

$$\hat{\Gamma}_k^{ij} = \frac{1}{3}\left(\Gamma_k^{ij} + \Gamma_k^{ji} + \frac{\partial\pi^{ji}}{\partial x^k} + {}^\dagger\Gamma_k^{ij}\right),$$

satisfy the condition

$$\pi^{ik} \hat{\Gamma}_k^{jl} = \pi^{jk} \hat{\Gamma}_k^{il},$$

which immediately follows from (6), (7), and the Jacobi identity for the Poisson tensor π^{ij}. $\quad\square$

On a symplectic manifold there exist symplectic (torsion-free) covariant connections. A symplectic covariant connection induces a torsion-free Poisson contravariant connection.

Another important example of torsion-free Poisson contravariant connections appears in the context of Kähler–Poisson manifolds. We call a complex manifold M Kähler–Poisson if it is endowed with a Poisson tensor $g^{\bar{l}k}$ of type $(1, 1)$ with respect to the complex structure. Here k and $\bar{l}$ are holomorphic and antiholomorphic indices, respectively, with respect to local holomorphic coordinates $\{z^k\}$ on M. The Jacobi identity for the Poisson tensor $g^{\bar{l}k}$ takes the form

$$g^{\bar{t}s} \frac{\partial g^{\bar{l}k}}{\partial z^s} = g^{\bar{l}s} \frac{\partial g^{\bar{t}k}}{\partial z^s} \qquad \text{and} \qquad g^{\bar{t}s} \frac{\partial g^{\bar{l}k}}{\partial \bar{z}^t} = g^{\bar{l}k} \frac{\partial g^{\bar{l}s}}{\partial \bar{z}^t}.$$

One can introduce a torsion-free Poisson contravariant connection $\nabla^\bullet$ on M with the following Christoffel symbols:

$$\Gamma_m^{\bar{l}k} = -\frac{\partial g^{\bar{l}k}}{\partial z^m}, \qquad \Gamma_{\bar{n}}^{k\bar{l}} = \frac{\partial g^{\bar{l}k}}{\partial \bar{z}^n} \tag{10}$$

(the symbols with the other types of indices are equal to zero). We call this contravariant connection Kähler–Poisson. If $g^{\bar{l}k}$ is nondegenerate, M becomes a Kähler manifold and the contravariant connection $\nabla^\bullet$ is induced by the Kähler connection.

We are going to show that the Poisson manifold which is the dual of a semisimple Lie algebra does not admit torsion-free Poisson contravariant connections. First we need to prove a technical lemma.

Lemma 5. *On a semisimple Lie algebra $\mathfrak{g}$ any linear mapping $Q : \mathfrak{g} \to \mathfrak{g}$ such that*

$$[X, Q(Y)] = [Y, Q(X)] \tag{11}$$

for all $X, Y \in \mathfrak{g}$ vanishes.

Proof. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear mapping satisfying (11). The Killing form $B(\cdot, \cdot)$ on $\mathfrak{g}$ is a nondegenerate invariant symmetric bilinear form. Thus $B([X, Y], Z) = B(X, [Y, Z])$, which implies that the function $\Theta(X, Y, Z) = B([X, Y], Q(Z)) = B(X, [Y, Q(Z)])$ is antisymmetric in (X, Y) and symmetric in (Y, Z). It can only happen if $\Theta = 0$. Since B is nondegenerate and for $\mathfrak{g}$ semisimple $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we obtain that $Q = 0$. $\quad\square$

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra with a fixed basis $\{X^i\}$ and the structure constants c_k^{ij} so that

$$[X^i, X^j] = c_k^{ij} X^k.$$

Denote by $\{x^i\}$ the corresponding linear coordinates on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. The standard linear Poisson structure on $\mathfrak{g}^*$ is given by the formula

$$\pi^{ij} = c_k^{ij} x^k. \tag{12}$$

Assume that $\nabla^\bullet$ is a torsion-free Poisson contravariant connection on $\mathfrak{g}^*$ with the Christoffel symbols Γ_k^{ij}. Thus

$$-\Gamma_k^{ij} + \Gamma_k^{ji} - \frac{\partial \pi^{ij}}{\partial x^k} = 0 \qquad \text{and} \qquad \pi^{ik}\Gamma_k^{jl} = \pi^{jk}\Gamma_k^{il}. \tag{13}$$

Formulas (12) and (13) imply that

$$-\Gamma_k^{ij}(0) + \Gamma_k^{ji}(0) - c_k^{ij} = 0 \qquad \text{and} \qquad c_s^{ik}\Gamma_k^{jl}(0) = c_s^{jk}\Gamma_k^{il}(0). \tag{14}$$

Introduce a bilinear mapping $\Gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$\Gamma(X^i, X^j) = -\Gamma_k^{ij}(0)X^k.$$

It follows from (14) that the mapping $\Gamma(X, Y)$ satisfies the following conditions:

$$\Gamma(X, Y) - \Gamma(Y, X) - [X, Y] = 0 \tag{15}$$

and

$$[X, \Gamma(Y, Z)] = [Y, \Gamma(X, Z)]. \tag{16}$$

If $\mathfrak{g}$ is a semisimple Lie algebra it follows from Lemma 5 and formula (16) that for any element $Z \in \mathfrak{g}$ the mapping $\mathfrak{g} \ni X \to \Gamma(X, Z)$ vanishes, which implies that $\Gamma = 0$. However, according to (15) the skew-symmetric part of $\Gamma(X, Y)$ is $\frac{1}{2}[X, Y]$. This contradiction shows that on the dual of a semisimple Lie algebra endowed with the standard linear Poisson structure there are no torsion-free Poisson contravariant connections. It is interesting to compare this statement with the theorem by Gutt and Rawnsley that the dual of a semisimple Lie algebra does not admit a tangential deformation quantization (see [6] and also [15]).

In the rest of the paper we will show that using Fedosov's approach one can give a geometric construction of a formal symplectic groupoid (see [10]) on an arbitrary Poisson manifold endowed with a torsion-free Poisson contravariant connection.

3. Nonlinear contravariant connections

The formal neighborhood (X, Y) of a submanifold Y of a manifold X is the ringed space on Y whose ring of global sections is $C^\infty(X)/\cap_{k=1}^\infty I_Y^k$, where I_Y denotes the ideal of functions in $C^\infty(M)$ vanishing on Y (see the Appendix to [9]). Let $p : E \to M$ be a vector bundle with a finite dimensional fibre. Denote by $\mathrm{Vect}(E, Z)$ the Lie algebra of formal vector fields on the formal neighborhood (E, Z) of the zero section Z of E. If M is a Poisson manifold we call a mapping $D^\bullet : \Omega^1(M) \to \mathrm{Vect}(E, Z)$ a contravariant connection on the formal neighborhood (E, Z) if it satisfies the axioms

$$D^{f\alpha} = p^*(f)D^\alpha, \qquad [D^\alpha, p^*f] = p^*((\#\alpha)f),$$

where $\alpha \in \Omega^1(M)$ and $f \in C^\infty(M)$.

Remark. If $D^\bullet$ leaves invariant the space of fibrewise linear functions on (E, Z), then it is induced by a linear connection on the dual bundle $E^* \to M$. In general, this is not the case and $D^\bullet$ will be referred to as a nonlinear contravariant connection on (E, Z). This object is analogous to an Ehresman connection.

We will be interested in the nonlinear contravariant connections on the formal neighborhood (T^*M, Z).

If $\nabla^\bullet$ is a linear contravariant connection on M with the Christoffel symbols Γ_k^{ij}, it induces a contravariant connection $\bar{\nabla}^\bullet$ on (T^*M, Z) expressed locally as

$$\bar{\nabla}^{dx^i} = \pi^{ij}\frac{\partial}{\partial x^j} + \Gamma_k^{ij}\xi_j\frac{\partial}{\partial \xi_k},$$

where $\{x^i\}$ are local coordinates on M lifted to (T^*M, Z) and $\{\xi_i\}$ are the dual fibre coordinates.

For a global vector field $v \in \text{Vect}(M)$ written locally as $v = v^i\partial_i$ denote by $\hat{v}$ the multiplication operator by the global function on $C^\infty(T^*M, Z)$ whose local expression is $v^i(x)\xi_i$. For a global 1-form $\alpha \in \Omega^1(M)$ written locally as $\alpha = \alpha_i dx^i$ denote by $\hat{\alpha}$ the global operator on $C^\infty(T^*M, Z)$ written locally as $\alpha_i\frac{\partial}{\partial \xi_i}$.

Define the torsion T^D of a nonlinear contravariant connection $D^\bullet$ on (T^*M, Z) as the skew-symmetric bilinear mapping from $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$ to the space $\text{Vect}_{\text{vert}}(T^*M, Z)$ of vertical vector fields on the formal neighborhood (T^*M, Z) given by the formula

$$T^D(\alpha, \beta) = \left[D^\alpha, \hat{\beta}\right] - \left[D^\beta, \hat{\alpha}\right] - \widehat{[\alpha, \beta]}.$$

It is a morphism of $C^\infty(M)$-modules. A nonlinear contravariant connection $D^\bullet$ on (T^*M, Z) can be written in local coordinates as follows:

$$D^{dx^i} = \pi^{is}\frac{\partial}{\partial x^s} - A_s^i\frac{\partial}{\partial \xi_s}, \tag{17}$$

where $A_s^i = A_s^i(x, \xi)$ is a formal function (the fibre coordinates $\{\xi_i\}$ are treated as formal variables). Locally

$$T^D(dx^i, dx^j) = \frac{\partial A_s^i}{\partial \xi_j}\frac{\partial}{\partial \xi_s} - \frac{\partial A_s^j}{\partial \xi_i}\frac{\partial}{\partial \xi_s} - \frac{\partial \pi^{ij}}{\partial x^s}\frac{\partial}{\partial \xi_s}. \tag{18}$$

One can check that the torsion T^∇ of a contravariant connection $\nabla^\bullet$ is related to the torsion $T^{\bar{\nabla}}$ of the induced contravariant connection $\bar{\nabla}^\bullet$ on (T^*M, Z) as follows:

$$T^{\bar{\nabla}}(\alpha, \beta) = \widehat{T^\nabla(\alpha, \beta)}$$

for any $\alpha, \beta \in \Omega^1(M)$.

A contravariant connection $\nabla^\bullet$ respects the Poisson bivector field Π if and only if the induced mapping $\bar{\nabla}^\bullet$ satisfies the equation

$$\left[\left[\bar{\nabla}^\gamma, \widehat{\#\alpha}\right], \hat{\beta}\right] = \left[\left[\bar{\nabla}^\gamma, \hat{\alpha}\right], \widehat{\#\beta}\right]$$

for any $\alpha, \beta, \gamma \in \Omega^1(M)$. We will call a nonlinear contravariant connection $D^\bullet$ Poisson if it satisfies the condition

$$\left[\left[D^\gamma, \widehat{\#\alpha}\right], \hat{\beta}\right] = \left[\left[D^\gamma, \hat{\alpha}\right], \widehat{\#\beta}\right] \tag{19}$$

for any $\alpha, \beta, \gamma \in \Omega^1(M)$.

A simple calculation shows that $D^\bullet$ written in local coordinates as in (17) satisfies (19) if and only if

$$\pi^{ms}\frac{\partial \pi^{ij}}{\partial x^s} - \frac{\partial A^m_s}{\partial \xi_i}\pi^{sj} - \frac{\partial A^m_s}{\partial \xi_j}\pi^{is} = 0. \tag{20}$$

Linear contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ are associated if and only if the mappings $\bar{\nabla}^\bullet, {}^\dagger\bar{\nabla}^\bullet : \Omega^1(M) \to \mathrm{Vect}(E, Z)$ satisfy the condition

$$\left[{}^\dagger\bar{\nabla}^\alpha, \widehat{\#\beta}\right] - \left[\bar{\nabla}^\beta, \widehat{\#\alpha}\right] - \widehat{\#[\alpha, \beta]} = 0. \tag{21}$$

We will say that nonlinear contravariant connections $(D^\bullet, {}^\dagger D^\bullet)$ satisfying

$$\left[{}^\dagger D^\alpha, \widehat{\#\beta}\right] - \left[D^\beta, \widehat{\#\alpha}\right] - \widehat{\#[\alpha, \beta]} = 0 \tag{22}$$

for any $\alpha, \beta \in \Omega^1(M)$ are associated. One can check that if, locally,

$$D^{\mathrm{d}x^i} = \pi^{is}\frac{\partial}{\partial x^s} - A^i_s\frac{\partial}{\partial \xi_s} \qquad \text{and} \qquad {}^\dagger D^{\mathrm{d}x^i} = \pi^{is}\frac{\partial}{\partial x^s} - K^i_s\frac{\partial}{\partial \xi_s}, \tag{23}$$

then (22) is equivalent to the condition

$$\pi^{is}A^j_s = \pi^{js}K^i_s. \tag{24}$$

Using formulas (18), (20) and (24) one can prove the following lemma.

Lemma 6. *If nonlinear contravariant connections $(D^\bullet, {}^\dagger D^\bullet)$ are associated and ${}^\dagger D^\bullet$ is torsion-free, then the nonlinear contravariant connection $D^\bullet$ is Poisson.*

The Poisson structure Π on M induces a fibrewise presymplectic form Ω on T^*M given locally by the formula

$$\Omega = \frac{1}{2}\pi^{ij}\,\mathrm{d}\xi_i \wedge \mathrm{d}\xi_j.$$

Denote by $\mathcal{H}_\Omega$ the subspace of $C^\infty(T^*M, Z)$ of Hamiltonian functions of the fibrewise presymplectic form Ω. Namely, $F \in \mathcal{H}_\Omega$ if there is a vertical formal vector field $H_F \in \mathrm{Vect}_{\mathrm{vert}}(T^*M, Z)$ such that

$$\iota(H_F)\Omega = -\mathrm{d}_\xi F,$$

where d_ξ is the fibrewise differential. The choice of H_F is, in general, not unique. If locally $H_F = a_i\frac{\partial}{\partial \xi_i}$, then

$$\frac{\partial F}{\partial \xi_i} = -\pi^{ji}a_j = \pi^{ij}a_j. \tag{25}$$

It follows from (25) that

$$\pi^{ij}\frac{\partial a_j}{\partial \xi_k} = \pi^{kj}\frac{\partial a_j}{\partial \xi_i}. \tag{26}$$

The space $\mathcal{H}_\Omega$ is a Poisson algebra with respect to the pointwise product and the bracket $\{\cdot, \cdot\}_\Omega$ defined as follows:

$$\{F, G\}_\Omega = H_F G = -H_G F = a_i \frac{\partial G}{\partial \xi_i} = \pi^{ij} a_i b_j, \tag{27}$$

where $F, G \in \mathcal{H}_\Omega$ and, locally, $H_F = a_i \frac{\partial}{\partial \xi_i}$ and $H_G = b_i \frac{\partial}{\partial \xi_i}$.

Remark. For any formal function $F \in C^\infty(TM, Z)$ its pullback $F^\#$ with respect to the mapping $\#$ belongs to $\mathcal{H}_\Omega$. For $F = F(x, y)$ we have $F^\#(x, \xi) = F(x, \xi_i \pi^{i\cdot})$. Therefore

$$\frac{\partial F^\#}{\partial \xi_i} = \pi^{ij} \left(\frac{\partial F}{\partial y^j} \right)^\#,$$

where $\{y^i\}$ are the local fibre coordinates on TM dual to $\{\xi_i\}$.

Proposition 2. *Given a Poisson nonlinear contravariant connection $D^\bullet$, the formal vector fields D^α, $\alpha \in \Omega^1(M)$, are derivations of the Poisson algebra $(\mathcal{H}_\Omega, \{\cdot, \cdot\}_\Omega)$.*

Proof. For $F \in \mathcal{H}_\Omega$ assume that locally $H_F = a_i \frac{\partial}{\partial \xi_i}$ and formula (25) holds. In order to prove the proposition we have to show that

$$b_t \frac{\partial}{\partial \xi_t} := [D^{dx^i}, H_F],$$

is a Hamiltonian vector field of the local Hamiltonian function $D^{dx^i} F$, i.e., that

$$\frac{\partial}{\partial \xi_j} \left(D^{dx^i} F \right) = \pi^{jt} b_t. \tag{28}$$

We have, using (25),

$$
\begin{aligned}
\frac{\partial}{\partial \xi_j} \left(D^{dx^i} F \right) &= \frac{\partial}{\partial \xi_j} \left(\pi^{is} \frac{\partial F}{\partial x^s} - A_s^i \frac{\partial F}{\partial \xi_s} \right) = \pi^{is} \frac{\partial^2 F}{\partial x^s \partial \xi_j} - \frac{\partial A_s^i}{\partial \xi_j} \frac{\partial F}{\partial \xi_s} \\
&\quad - A_s^i \frac{\partial^2 F}{\partial \xi_s \partial \xi_j} = \pi^{is} \frac{\partial}{\partial x^s} \left(\pi^{jt} a_t \right) - \frac{\partial A_s^i}{\partial \xi_j} \pi^{st} a_t - A_s^i \pi^{jt} \frac{\partial a_t}{\partial \xi_s} \\
&= \pi^{is} \pi^{jt} \frac{\partial a_t}{\partial x^s} + \pi^{is} \frac{\partial \pi^{jt}}{\partial x^s} a_t - \frac{\partial A_s^i}{\partial \xi_j} \pi^{st} a_t - A_s^i \pi^{jt} \frac{\partial a_t}{\partial \xi_s}.
\end{aligned}
\tag{29}
$$

On the other hand we obtain that

$$
\begin{aligned}
b_t \frac{\partial}{\partial \xi_t} &= \left[\pi^{is} \frac{\partial}{\partial x^s} - A_s^i \frac{\partial}{\partial \xi_s}, a_t \frac{\partial}{\partial \xi_t} \right] \\
&= \pi^{is} \frac{\partial a_t}{\partial x^s} \frac{\partial}{\partial \xi_t} - A_s^i \frac{\partial a_t}{\partial \xi_s} \frac{\partial}{\partial \xi_t} + \frac{\partial A_t^i}{\partial \xi_k} a_k \frac{\partial}{\partial \xi_t}.
\end{aligned}
\tag{30}
$$

Taking into account that $D^\bullet$ is a Poisson connection we obtain formula (28) from (20), (29) and (30), whence the proposition follows. $\square$

4. Lifting functions via formal nonlinear connections

There is a natural grading deg on the space $C^\infty(T^*M, Z)$ by the powers of the formal fibre variables ξ_i. It induces a grading on the endomorphisms of $C^\infty(T^*M, Z)$. Thus $\deg(\xi_i) = 1$ and $\deg\left(\frac{\partial}{\partial \xi_i}\right) = -1$. For a linear contravariant connection $\nabla^\bullet$ on M and a 1-form $\alpha \in \Omega^1(M)$ we have $\deg\left(\bar\nabla^\alpha\right) = 0$. A nonlinear contravariant connection $D^\bullet$ can be expanded into a ξ-adically convergent series

$$D^\bullet = D^\bullet_{-1} + D^\bullet_0 + \cdots,$$

where $D^\bullet_d$ is the homogeneous component of $D^\bullet$ of degree d. There is a unique contravariant connection $\nabla^\bullet$ such that $D^\bullet_0 = \bar\nabla^\bullet$. For any $d \neq 0$ the component $D^\bullet_d$ is a mapping from $\Omega^1(M)$ to $\mathrm{Vect}_{\mathrm{vert}}(T^*M, Z)$. The mapping $\alpha \mapsto D^\alpha_{-1}$ determines uniquely a linear endomorphism ψ_D of the cotangent bundle T^*M such that

$$D^\alpha_{-1} = \widehat{\psi_D(\alpha)}$$

for all $\alpha \in \Omega^1(M)$. We will call a nonlinear contravariant connection $D^\bullet$ *invertible* if ψ_D is invertible. We will give necessary and sufficient conditions on an invertible nonlinear contravariant connection $D^\bullet$ under which the system

$$D^\bullet F = 0 \tag{31}$$

with the initial condition $F|_Z = p^*(f)|_Z$ has a unique solution $F \in C^\infty(T^*M, Z)$ for an arbitrary $f \in C^\infty(M)$.

The curvature of a nonlinear contravariant connection $D^\bullet$ is a skew-symmetric morphism of $C^\infty(M)$-modules

$$R^D : \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M) \to \mathrm{Vect}_{\mathrm{vert}}(T^*M, Z)$$

such that for $\alpha, \beta \in \Omega^1(M)$

$$R^D(\alpha, \beta) = \left[D^\alpha, D^\beta\right] - D^{[\alpha,\beta]}.$$

If a nonlinear contravariant connection $D^\bullet$ is written locally as in (17), then

$$R^D(dx^i, dx^j) = \left[D^{dx^i}, D^{dx^j}\right] - \frac{\partial \pi^{ij}}{\partial x^k} D^{dx^k} = R^{ij}_k \frac{\partial}{\partial \xi_k}, \tag{32}$$

where

$$R^{ij}_k = -D^{dx^i}\left(A^j_k\right) + D^{dx^j}\left(A^i_k\right) + \frac{\partial \pi^{ij}}{\partial x^s} A^s_k. \tag{33}$$

System (31) written locally takes the form

$$\pi^{is} \frac{\partial F}{\partial x^s} = A^i_s \frac{\partial F}{\partial \xi_s}. \tag{34}$$

Since $D^\bullet$ is invertible, the matrix $\left(A^i_j\right)$ is formally invertible. Denote by $\left(B^i_j\right)$ its inverse. System (34) is equivalent to the following one:

$$\frac{\partial F}{\partial \xi_i} = B^i_j \pi^{js} \frac{\partial F}{\partial x^s}. \tag{35}$$

It is well known that system (35) has a unique formal solution $F(x, \xi)$ with the initial condition $F(x, 0) = f(x)$ for an arbitrary smooth function $f(x)$ if and only if the operators

$$B_j^i D^{\mathrm{d}x^j} = B_j^i \pi^{js} \frac{\partial}{\partial x^s} - \frac{\partial}{\partial \xi_i}$$

pairwise commute. We have, taking into account the identity

$$D^{\mathrm{d}x^k} B_l^j = -B_p^j D^{\mathrm{d}x^k} \left(A_q^p \right) B_l^q$$

and formulas (32) and (33), that

$$\begin{aligned}
\left[B_k^i D^{\mathrm{d}x^k}, B_l^j D^{\mathrm{d}x^l} \right] &= B_k^i D^{\mathrm{d}x^k} \left(B_l^j \right) D^{\mathrm{d}x^l} - B_k^j D^{\mathrm{d}x^k} \left(B_l^i \right) D^{\mathrm{d}x^l} + B_k^i B_l^j \left[D^{\mathrm{d}x^k}, D^{\mathrm{d}x^l} \right] \\
&= B_k^i B_l^j \left(-D^{\mathrm{d}x^k} \left(A_q^l \right) B_p^q D^{\mathrm{d}x^p} + D^{\mathrm{d}x^l} \left(A_q^k \right) B_p^q D^{\mathrm{d}x^p} + \left[D^{\mathrm{d}x^k}, D^{\mathrm{d}x^l} \right] \right) \\
&= B_k^i B_l^j \left(\left(R_q^{kl} - \frac{\partial \pi^{kl}}{\partial x^s} A_q^s \right) B_p^q D^{\mathrm{d}x^p} + \left[D^{\mathrm{d}x^k}, D^{\mathrm{d}x^l} \right] \right) \\
&= B_k^i B_l^j \left(R_q^{kl} B_p^q D^{\mathrm{d}x^p} + R_q^{kl} \frac{\partial}{\partial \xi_q} \right) = B_k^i B_l^j R_q^{kl} B_p^q \pi^{ps} \frac{\partial}{\partial x^s}.
\end{aligned}$$

Since the matrix $\left(B_j^i \right)$ is invertible, we obtain the following proposition.

Proposition 3. *For an invertible nonlinear contravariant connection $D^\bullet$ written locally as (17) system (31) with the initial condition $F(x, 0) = f(x)$ has a unique solution $F = F(x, \xi)$ for an arbitrary smooth function $f(x)$ if and only if*

$$R_q^{kl} B_p^q \pi^{ps} = 0, \tag{36}$$

where R_q^{kl} are the coefficients of the curvature of the connection $D^\bullet$ given by formula (33) and $\left(B_j^i \right)$ is the inverse of $\left(A_j^i \right)$.

Proposition 3 gives only a local necessary and sufficient condition of the existence and uniqueness of solutions of system (31). To give a global criterion, we assume additionally that the invertible connection $D^\bullet$ has an associated invertible nonlinear contravariant connection $^\dagger D^\bullet$. Suppose that $D^\bullet$ and $^\dagger D^\bullet$ are written locally as in (23). Since $D^\bullet$ and $^\dagger D^\bullet$ are invertible, there exist the inverse matrices (B_j^i) and (L_j^i) of (A_j^i) and (K_j^i), respectively. It follows from (24) that

$$\pi^{is} B_s^j = \pi^{js} L_s^i. \tag{37}$$

Then, if R_q^{kl} are the coefficients of the curvature R^D, condition (36) is equivalent to the following one:

$$R_q^{kl} \pi^{qs} L_s^p = 0. \tag{38}$$

Since (L_j^i) is invertible, we obtain that (38) is equivalent to the condition

$$R_q^{kl} \pi^{qs} = 0$$

which means that the vertical formal vector field $R^D(\mathrm{d}x^k, \mathrm{d}x^l)$ annihilates the fibrewise presymplectic form Ω. Thus we arrive at the following theorem.

Theorem 1. *Let $(D^\bullet, {}^\dagger D^\bullet)$ be a pair of associated invertible nonlinear formal contravariant connections. Then system* (31) *with the initial condition $F|_Z = p^*(f)|_Z$ has a unique solution $F \in C^\infty(T^*M, Z)$ for an arbitrary $f \in C^\infty(M)$ if and only if for any $\alpha, \beta \in \Omega^1(M)$ the vertical vector field $R^D(\alpha, \beta)$ is in the kernel of the fibrewise presymplectic form Ω,*

$$\iota\left(R^D(\alpha, \beta)\right)\Omega = 0.$$

Another important property of such a pair of contravariant connections is given by the following lemma.

Lemma 7. *Let $(D^\bullet, {}^\dagger D^\bullet)$ be a pair of associated invertible nonlinear formal contravariant connections. Then the kernel of the connection $D^\bullet$ is a subspace of $\mathcal{H}_\Omega$.*

Proof. Write the connections $D^\bullet$ and ${}^\dagger D^\bullet$ locally as in (23) and let (B^i_j) and (L^i_j) be the inverse matrices of (A^i_j) and (K^i_j), respectively. If $F \in C^\infty(T^*M, Z)$ is a solution of system (31), then F is also a solution of system (35). It follows from formula (37) that

$$\frac{\partial F}{\partial \xi_i} = -\pi^{ij}L^s_j\frac{\partial F}{\partial x^s},$$

which implies that $F \in \mathcal{H}_\Omega$. $\quad\square$

In general, system (31) can have the same space of solutions for different nonlinear contravariant connections $D^\bullet$. Given two pairs of associated invertible nonlinear contravariant connections, $(D^\bullet_1, {}^\dagger D^\bullet_1)$ and $(D^\bullet_2, {}^\dagger D^\bullet_2)$, one can show that if for any $\alpha, \beta \in \Omega^1(M)$

$$\left[D^\alpha_1, \widehat{\#\beta}\right] = \left[D^\alpha_2, \widehat{\#\beta}\right],$$

then the kernels of the connections $D^\bullet_1, D^\bullet_2$ coincide.

5. The fundamental equation

In this section we will assume that $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ are associated global contravariant connections with ${}^\dagger\nabla^\bullet$ torsion-free on a Poisson manifold (M, Π). Starting with these connections, we are going to construct a global Poisson morphism $\theta : (C^\infty(M), \{\cdot, \cdot\}) \to (\mathcal{H}_\Omega, \{\cdot, \cdot\}_\Omega)$.

Consider local nonlinear contravariant connections

$$D^{\mathrm{d}x^i} = \pi^{is}\frac{\partial}{\partial x^s} + \Gamma^{ij}_s \xi_j\frac{\partial}{\partial \xi_s} - v^i_s\frac{\partial}{\partial \xi_s}, \tag{39}$$

$$^\dagger D^{\mathrm{d}x^i} = \pi^{is}\frac{\partial}{\partial x^s} + {}^\dagger\Gamma^{ij}_s \xi_j\frac{\partial}{\partial \xi_s} - w^i_s\frac{\partial}{\partial \xi_s}, \tag{40}$$

where $\Gamma^{ij}_k, {}^\dagger\Gamma^{ij}_k$ are the Christoffel symbols of the connections $\nabla^\bullet, {}^\dagger\nabla^\bullet$, respectively, and $v^i_k = v^i_k(x, \xi), w^i_k = w^i_k(x, \xi)$ are formal functions.

It follows from formulas (21) and (22) that connections $D^\bullet$ and ${}^\dagger D^\bullet$ are associated if and only if

$$\pi^{is}v^j_s = \pi^{js}w^i_s.$$

Connection $^\dagger D^\bullet$ is torsion-free if and only if

$$\frac{\partial w_k^i}{\partial \xi_j} - \frac{\partial w_k^j}{\partial \xi_i} = 0,$$

or, equivalently, if there exists a formal function $u_k(x, \xi)$ such that

$$w_k^i = \frac{\partial u_k}{\partial \xi_i}.$$

Now assume that $D^\bullet$ and $^\dagger D^\bullet$ are associated and $^\dagger D^\bullet$ is torsion-free. We will write

$$^\dagger D^{\mathrm{d}x^i} = \pi^{is} \frac{\partial}{\partial x^s} + {}^\dagger \Gamma_s^{ij} \, \xi_j \frac{\partial}{\partial \xi_s} - \frac{\partial u_s}{\partial \xi_i} \frac{\partial}{\partial \xi_s}. \tag{41}$$

The condition that $D^\bullet$ and $^\dagger D^\bullet$ are associated takes the form

$$\pi^{is} v_s^j = \pi^{js} \frac{\partial u_s}{\partial \xi_i} = \frac{\partial \left(\pi^{js} u_s \right)}{\partial \xi_i}. \tag{42}$$

It follows from (42) that $\pi^{js} u_s$ is a local Hamiltonian function for the fibrewise presymplectic structure defined by Ω.

The coefficients $\bar{R}_k^{ij}$ of the curvature $R^{\bar{\nabla}}$ of the mapping $\bar{\nabla}^\bullet$ are given by the formula

$$\bar{R}_k^{ij} = \pi^{is} \frac{\partial \Gamma_k^{jp}}{\partial x^s} \xi_p - \pi^{js} \frac{\partial \Gamma_k^{ip}}{\partial x^s} \xi_p - \frac{\partial \pi^{ij}}{\partial x^s} \Gamma_k^{sp} \xi_p - \Gamma_k^{iq} \Gamma_q^{jp} \xi_p + \Gamma_k^{jq} \Gamma_q^{ip} \xi_p. \tag{43}$$

Denote by $\mathcal{E}$ the fibrewise Euler vector field on T^*M expressed locally as

$$\mathcal{E} = \xi_k \frac{\partial}{\partial \xi_k}$$

and introduce a skew-symmetric bilinear morphism of $C^\infty(M)$-modules

$$\bar{Q} : \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M) \to C^\infty(T^*M)$$

as follows:

$$\bar{Q}(\alpha, \beta) = \frac{1}{2} \Omega \left(\mathcal{E}, R^{\bar{\nabla}}(\alpha, \beta) \right).$$

Introduce a function $\bar{Q}^{ij} = \bar{Q}(\mathrm{d}x^i, \mathrm{d}x^j)$. Then

$$\bar{Q}^{ij} = \frac{1}{2} \xi_t \pi^{tk} \bar{R}_k^{ij}. \tag{44}$$

Using formulas (6), (7) and (44), and the Jacobi identity for the Poisson tensor π^{ij} one can show that

$$\frac{\partial}{\partial \xi_p} \left(\pi^{tk} \bar{R}_k^{ij} \right) = \frac{1}{2} \pi^{is} \pi^{jk} \left(\frac{\partial}{\partial x^s} \left({}^\dagger \Gamma_k^{tp} + {}^\dagger \Gamma_k^{pt} \right) - \frac{\partial}{\partial x^k} \left({}^\dagger \Gamma_s^{tp} + {}^\dagger \Gamma_s^{pt} \right) \right)$$
$$+ \Gamma_k^{it} \pi^{kl} \Gamma_l^{jp} + \Gamma_k^{ip} \pi^{kl} \Gamma_l^{jt}. \tag{45}$$

The right-hand side of (45) is symmetric with respect to the permutation of the indices p and t. It follows from (44) and (45) that

$$\pi^{tk} \bar{R}_k^{ij} = \frac{\partial}{\partial \xi_t} \bar{Q}^{ij}. \tag{46}$$

Formula (46) means that for any indices i, j the function $\bar{Q}^{ij}$ is a local Hamiltonian function for the presymplectic structure defined by Ω. This can be written globally as follows:

$$\iota \left(R^{\bar{\nabla}}(\alpha, \beta) \right) \Omega = -d_\xi \bar{Q}(\alpha, \beta).$$

The coefficients R_k^{ij} of the curvature R^D of the connection $D^\bullet$ are expressed as

$$
\begin{aligned}
R_k^{ij} = {}& \bar{R}_k^{ij} - \pi^{is} \frac{\partial v_k^j}{\partial x^s} + \pi^{js} \frac{\partial v_k^i}{\partial x^s} + \frac{\partial \pi^{ij}}{\partial x^s} v_k^s - \Gamma_s^{ip} \xi_p \frac{\partial v_k^j}{\partial \xi_s} \\
& + \Gamma_s^{jp} \xi_p \frac{\partial v_k^i}{\partial \xi_s} + \Gamma_k^{ip} v_p^j - \Gamma_k^{jp} v_p^i + v_s^i \frac{\partial v_k^j}{\partial \xi_s} - v_s^j \frac{\partial v_k^i}{\partial \xi_s}.
\end{aligned} \tag{47}
$$

We want to show that there is a function Q^{ij} such that

$$\pi^{tk} R_k^{ij} = \frac{\partial}{\partial \xi_t} Q^{ij}. \tag{48}$$

Taking into account Lemma 3 we get from (47) that

$$
\begin{aligned}
\pi^{tk} R_k^{ij} = {}& \pi^{tk} \bar{R}_k^{ij} - \pi^{is} \frac{\partial \left(\pi^{tk} v_k^j \right)}{\partial x^s} + \pi^{js} \frac{\partial \left(\pi^{tk} v_k^i \right)}{\partial x^s} + \frac{\partial \pi^{ij}}{\partial x^s} \pi^{tk} v_k^s \\
& - \Gamma_s^{ip} \xi_p \frac{\partial \left(\pi^{tk} v_k^j \right)}{\partial \xi_s} + \Gamma_s^{jp} \xi_p \frac{\partial \left(\pi^{tk} v_k^i \right)}{\partial \xi_s} - \Gamma_s^{it} \pi^{sk} v_k^j + \Gamma_s^{jt} \pi^{sk} v_k^i \\
& + v_s^i \pi^{tk} \frac{\partial v_k^j}{\partial \xi_s} - v_s^j \pi^{tk} \frac{\partial v_k^i}{\partial \xi_s}.
\end{aligned} \tag{49}
$$

Using formulas (26), (42) and (46) we obtain from (49) that

$$
\begin{aligned}
\pi^{tk} R_k^{ij} = {}& \frac{\partial}{\partial \xi_t} \left(\bar{Q}^{ij} - \pi^{is} \frac{\partial \left(\pi^{jk} u_k \right)}{\partial x^s} + \pi^{js} \frac{\partial \left(\pi^{ik} u_k \right)}{\partial x^s} + \frac{\partial \pi^{ij}}{\partial x^s} \pi^{sk} u_k \right. \\
& \left. - \Gamma_s^{ip} \xi_p \frac{\partial \left(\pi^{jk} u_k \right)}{\partial \xi_s} + \Gamma_s^{jp} \xi_p \frac{\partial \left(\pi^{ik} u_k \right)}{\partial \xi_s} + \pi^{st} v_s^i v_t^j \right).
\end{aligned} \tag{50}
$$

Formulas (27) and (50) imply that (48) holds for

$$
\begin{aligned}
Q^{ij} = {}& \bar{Q}^{ij} - \bar{\nabla}^i \left(\pi^{jk} u_k \right) + \bar{\nabla}^j \left(\pi^{ik} u_k \right) \\
& + \frac{\partial \pi^{ij}}{\partial x^s} \pi^{sk} u_k + \{ \pi^{ik} u_k, \pi^{jl} u_l \}_\Omega.
\end{aligned} \tag{51}
$$

Let U be a global vertical 1-form on (T^*M, Z). For a 1-form $\alpha \in \Omega^1(M)$ define an element $U(\alpha) \in C^\infty(T^*M, Z)$ as follows:

$$U(\alpha) = \langle U, \hat{\alpha} \rangle.$$

The mapping $\Omega^1(M) \ni \alpha \mapsto U(\alpha)$ from $\Omega^1(M)$ to $C^\infty(T^*M, Z)$ is a global morphism of $C^\infty(M)$-modules. Now assume that U is a global vertical 1-form on (T^*M, Z) for which there exist local potentials $u_k(x, \xi)$ and $v_k^i(x, \xi)$ (on a covering by coordinate charts) satisfying (42) and such that

$$U(\mathrm{d}x^i) = \pi^{ik}u_k, \tag{52}$$

$$u_k = -\xi_k (\mathrm{mod}\ \xi^2), \qquad \text{and} \qquad v_k^i = \delta_k^i (\mathrm{mod}\ \xi). \tag{53}$$

We see from (42) and (52) that $U(\alpha) \in \mathcal{H}_\Omega$ for any $\alpha \in \Omega^1(M)$.

Introduce *local* contravariant connections $D^\bullet$ and $^\dagger D^\bullet$ by formulas (39) and (41), respectively. Thus $^\dagger D^\bullet$ is torsion-free. It follows from (42) that $D^\bullet$ and $^\dagger D^\bullet$ are associated. Conditions (53) imply that both $D^\bullet$ and $^\dagger D^\bullet$ are invertible. Lemma 7 means that system (31) with $F \in C^\infty(T^*M, Z)$ has the same solutions as system (31) with $F \in \mathcal{H}_\Omega$. It follows from (27) and (42) that if $F \in \mathcal{H}_\Omega$, system (31) can be rewritten as

$$\bar{\nabla}^\alpha F = \{U(\alpha), F\}_\Omega. \tag{54}$$

Introduce a global skew-symmetric mapping

$$Q : \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M) \to C^\infty(T^*M, Z)$$

by the formula

$$\begin{aligned}
Q(\alpha, \beta) = {} & \bar{Q}(\alpha, \beta) - \nabla^\alpha U(\beta) + \nabla^\beta U(\alpha) \\
& + U([\alpha, \beta]) + \{U(\alpha), U(\beta)\}_\Omega.
\end{aligned} \tag{55}$$

It is easy to check that Q is a morphism of $C^\infty(M)$-modules. Notice that

$$Q(\mathrm{d}x^i, \mathrm{d}x^j) = Q^{ij},$$

where Q^{ij} is given by formula (51). It follows from formula (48) that

$$\iota\left(R^D(\alpha, \beta)\right)\Omega = -\mathrm{d}_\xi Q(\alpha, \beta). \tag{56}$$

We obtain from Theorem 1 and (56) that for any $f \in C^\infty(M)$ system (54) has a unique solution $F \in \mathcal{H}_\Omega$ such that $F|_Z = p^*(f)$ if and only if

$$\mathrm{d}_\xi Q(\alpha, \beta) = 0. \tag{57}$$

Taking into account conditions (53) we get from (57) that

$$Q(\alpha, \beta) = \Pi(\alpha, \beta).$$

Summing up the results obtained in this section we can state the following theorem.

Theorem 2. *If U is a global vertical 1-form on (T^*M, Z) which has local potentials $u_k(x, \xi)$ and $v_k^i(x, \xi)$ such that conditions (42), (52) and (53) hold and which satisfies the equation*

$$\Pi(\alpha, \beta) = \bar{Q}(\alpha, \beta) - \bar{\nabla}^{\alpha} U(\beta) + \bar{\nabla}^{\beta} U(\alpha)$$
$$+ U([\alpha, \beta]) + \{U(\alpha), U(\beta)\}_{\Omega}, \tag{58}$$

then for any $f \in C^{\infty}(M)$ *system* (54) *has a unique solution* $F \in \mathcal{H}_{\Omega}$ *such that* $F|_Z = p^*(f)$.

We call Eq. (58) the Fundamental Equation. If U is as in Theorem 2, introduce the mapping

$$\theta : C^{\infty}(M) \to \mathcal{H}_{\Omega}$$

which maps $f \in C^{\infty}(M)$ to the solution $F \in \mathcal{H}_{\Omega}$ of system (54) with the initial condition $F|_Z = p^*(f)$. We get from Lemma 6 and Proposition 2 that the mapping

$$\mathcal{H}_{\Omega} \ni F \mapsto \bar{\nabla}^{\alpha} F - \{U(\alpha), F\}_{\Omega}$$

is a derivation of the Poisson algebra $(\mathcal{H}_{\Omega}, \{\cdot, \cdot\}_{\Omega})$, whence we see that the solutions of system (54) are closed with respect to the Poisson bracket $\{\cdot, \cdot\}_{\Omega}$. Using this fact one can readily show that the mapping θ is a Poisson morphism.

Proposition 4. *If U is a global vertical 1-form on (T^*M, Z) which has local potentials $u_k(x, \xi)$ and $v_k^i(x, \xi)$ such that conditions (42), (52) and (53) hold and which satisfies the Fundamental Equation, the corresponding mapping θ is a Poisson morphism from $(C^{\infty}(M), \{\cdot, \cdot\})$ to $(\mathcal{H}_{\Omega}, \{\cdot, \cdot\}_{\Omega})$.*

The following theorem which states the existence and uniqueness of a normalized solution of the Fundamental Equation can be proved by induction.

Theorem 3. *Given a pair $(\nabla^{\bullet}, {}^{\dagger}\nabla^{\bullet})$ of associated contravariant connections on M such that ${}^{\dagger}\nabla^{\bullet}$ is torsion-free, there exists a unique global solution U of the Fundamental Equation (58) which satisfies the normalization condition*

$$\langle U, \mathcal{E} \rangle = 0 \tag{59}$$

and has local potentials $u_k(x, \xi)$ and $v_k^i(x, \xi)$ such that conditions (42), (52) and (53) hold.

We will give a recurrence formula for

$$u^i := U(\mathrm{d}x^i) \tag{60}$$

which proves the uniqueness part of the theorem. For a function $\alpha(x, \xi) \in C^{\infty}(T^*M, Z)$ denote by $\alpha^{(s)}$ its homogeneous component of degree s with respect to the fibre variables ξ. Assume that U is a normalized solution of the Fundamental Equation. Locally normalization condition (59) reads

$$u^i \xi_i = 0, \tag{61}$$

where u^i is given by (60). The Fundamental Equation can be written locally as follows:

$$\pi^{ij} = \bar{Q}^{ij} - \bar{\nabla}^{\mathrm{d}x^i} u^j + \bar{\nabla}^{\mathrm{d}x^j} u^i + \frac{\partial \pi^{ij}}{\partial x^k} u^k + \{u^i, u^j\}_{\Omega}. \tag{62}$$

Taking into account formulas (27), (44) and (53), observe that $\bar{Q}^{ij}$ is quadratic in ξ, the Poisson bracket $\{\cdot, \cdot\}_{\Omega}$ reduces the degree of homogeneity by two, $(u^i)^{(1)} = -\pi^{ij}\xi_j$, and for $F \in \mathcal{H}_{\Omega}$

$$\{(u^i)^{(1)}, F\}_{\Omega} = \{-\pi^{ij}\xi_j, F\}_{\Omega} = \frac{\partial F}{\partial \xi_i}. \tag{63}$$

Extracting from (62) the homogeneous component of degree $s - 1$, $s \geq 2$, we obtain, using (63), the following formula:

$$\frac{\partial (u^i)^{(s)}}{\partial \xi_j} - \frac{\partial (u^j)^{(s)}}{\partial \xi_i} = \alpha_s^{ij}, \tag{64}$$

where

$$\alpha_s^{ij} = \left(\bar{Q}^{ij} \right)^{(s-1)} - \bar{\nabla}^{\mathrm{d}x^i} \left(u^j \right)^{(s-1)} + \bar{\nabla}^{\mathrm{d}x^j} \left(u^i \right)^{(s-1)}$$

$$+ \frac{\partial \pi^{ij}}{\partial x^k} \left(u^k \right)^{(s-1)} + \sum_{t=1}^{s-2} \{ (u^i)^{(t+1)}, (u^j)^{(s-t)} \}_\Omega. \tag{65}$$

Taking into account normalization condition (61), we find a unique solution of (64) which provides a recurrence formula for u^i:

$$(u^i)^{(s)} = \frac{1}{s+1} \alpha_s^{ij} \xi_j. \tag{66}$$

Remark. The existence part of the proof of Theorem 3 involves a simultaneous recursive calculation, along with u^i, of the local potentials u_k and v_k^i satisfying (42) and the condition

$$u^i = \pi^{ik} u_k.$$

This calculation is based upon the following observations. The function α_s^{ij} can be represented in the form

$$\alpha_s^{ij} = \pi^{ik} A_{s;k}^j, \tag{67}$$

where

$$A_{s;k}^j = (\bar{A}_k^j)^{(s-1)} - \frac{\partial (u^j)^{(s-1)}}{\partial x^k} + {}^\dagger \Gamma_k^{ml} \xi_l (v_m^j)^{(s-2)} + \pi^{jl} \frac{\partial u_k^{(s-1)}}{\partial x^l}$$

$$+ \Gamma_l^{jm} \xi_m \frac{\partial u_k^{(s-1)}}{\partial x^l} + \frac{\partial \pi^{jl}}{\partial x^k} u_l^{(s-1)} - \sum_{t=1}^{s-2} \frac{\partial (u^l)^{(t+1)}}{\partial \xi_k} (v_l^j)^{(s-t+1)},$$

and

$$\bar{A}_k^j = \frac{1}{2} \xi_p \xi_q \pi^{jm} \left(\frac{\partial {}^\dagger \Gamma_m^{pq}}{\partial x^k} - \frac{\partial {}^\dagger \Gamma_k^{pq}}{\partial x^m} \right) - \xi_p \xi_q \, {}^\dagger \Gamma_k^{lp} \, \Gamma_l^{jq}.$$

This provides a recurrence formula for u_k:

$$u_k^{(s)} = \frac{1}{s+1} A_{s;k}^j \xi_j.$$

On the one hand, to obtain a recurrence formula for v_k^i we have to satisfy the condition

$$\frac{\partial (u^i)^{(s)}}{\partial \xi_j} = \pi^{jk} (v_k^i)^{(s-1)}.$$

On the other hand, it follows from (66) that

$$\frac{\partial (u^i)^{(s)}}{\partial \xi_j} = \frac{1}{s+1} \left(\alpha_s^{ij} + \frac{\partial \alpha_s^{il}}{\partial \xi_j} \xi_l \right).$$

Formulas (67) and $\alpha_s^{ij} = -\alpha_s^{ji}$ imply that

$$\alpha_s^{ij} = \pi^{jk}(-A_{s;k}^i).$$

One can check that

$$\frac{\partial \alpha_s^{il}}{\partial \xi_j} = \pi^{jk} B_{s;k}^{il},$$

where

$$\begin{aligned}
B_{s;k}^{il} = {} & \left(\bar{R}_k^{il} \right)^{(s-2)} - \pi^{im} \frac{\partial (v_k^l)^{(s-2)}}{\partial x^m} + \pi^{lm} \frac{\partial (v_k^i)^{(s-2)}}{\partial x^m} \\
& - \Gamma_n^{im} \xi_m \frac{\partial (v_k^l)^{(s-2)}}{\partial \xi_n} + \Gamma_n^{lm} \xi_m \frac{\partial (v_k^i)^{(s-2)}}{\partial \xi_n} \\
& + \Gamma_k^{im} (v_m^l)^{(s-2)} - \Gamma_k^{lm} (v_m^i)^{(s-2)} + \frac{\partial \pi^{il}}{\partial x^m} (v_k^m)^{(s-2)} \\
& + \sum_{t=1}^{s-2} \left(-\frac{\partial (v_k^i)^{(t)}}{\partial \xi_m} (v_m^l)^{(s-t-1)} + (v_m^i)^{(t)} \frac{\partial (v_k^l)^{(s-t-1)}}{\partial \xi_m} \right).
\end{aligned}$$

Thus we can give a recurrence formula for v_k^i:

$$(v_k^i)^{(s-1)} = \frac{1}{s+1} \left(-A_{s;k}^i + B_{s;k}^{il} \xi_l \right).$$

Next we want to give a recurrence formula for the unique global solution $F \in \mathcal{H}_\Omega$ of system (54) with the initial condition $F^{(0)} = f$, $f \in C^\infty(M)$. Extracting from the equation

$$\bar{\nabla}^{\mathrm{d}x^i} F = \{u^i, F\}_\Omega$$

its homogeneous component of degree $s-1$, $s \geq 1$, we obtain, using (63), the following formula:

$$\frac{\partial F^{(s)}}{\partial \xi_i} = \beta_s^i, \tag{68}$$

where

$$\beta_s^i = \bar{\nabla}^{\mathrm{d}x^i} F^{(s-1)} - \sum_{t=1}^{s-1} \{(u^i)^{(t+1)}, F^{(s-t)}\}_\Omega. \tag{69}$$

Thus we get a recurrence formula for F:

$$F^{(s)} = \frac{1}{s} \beta_s^i \xi_i. \tag{70}$$

6. Fedosov's formal symplectic groupoids

Assume that on a Poisson manifold M with the Poisson tensor π^{ij} there is a pair of associated global contravariant connections $(\nabla^\bullet, {}^\dagger\nabla^\bullet)$ with ${}^\dagger\nabla^\bullet$ torsion-free, and on (T^*M, Z) there is a global vertical 1-form U with local potentials u_k and v_k^i as in Theorem 2. These local potentials define associated local nonlinear invertible contravariant connections $D^\bullet$ and ${}^\dagger D^\bullet$ given by formulas (39) and (41). Thus ${}^\dagger D^\bullet$ is torsion-free. Assume also that there exist global tensors P_j^i and Q_j^i on M such that the contravariant connection $\nabla^\bullet$ respects them,

$$P_j^i + Q_j^i = \delta_j^i, \qquad \text{and} \qquad \pi^{ik} P_k^j = Q_k^i \pi^{kj}. \tag{71}$$

We give two basic examples of the tensors P_j^i and Q_j^i. The first one is $P_j^i = Q_j^i = (1/2)\delta_j^i$. The second example is related to the case of a Kähler–Poisson manifold endowed with the Kähler–Poisson contravariant connection (see Section 2). In this case one can set

$$P_m^k = \delta_m^k \qquad \text{and} \qquad Q_{\bar n}^{\bar l} = \delta_{\bar n}^{\bar l}, \tag{72}$$

where k, m are holomorphic and $\bar l, \bar n$ antiholomorphic indices. The tensor coefficients with mixed types of indices are set to zero. Otherwise speaking, P projects a tangent vector onto its component of type $(1, 0)$ and Q onto that of type $(0, 1)$.

Define a local change of variables on (T^*M, Z), $(x, \zeta) \mapsto (x, \xi)$ such that

$$\xi_p = u_p\left(x, -\zeta_j P_\cdot^j\right) - u_p\left(x, \zeta_j Q_\cdot^j\right). \tag{73}$$

We see from (53) and (71) that $\zeta_p = \xi_p (\text{mod } \xi^2)$, thus the change of variables is invertible. Introduce local mappings $f \mapsto Sf$, $f \mapsto Tf$ from functions on M to functions on (T^*M, Z) as follows:

$$(Sf)(x, \xi) = \theta(f)\left(x, -\zeta_j P_\cdot^j\right) \qquad \text{and} \qquad (Tf)(x, \xi) = \theta(f)\left(x, \zeta_j Q_\cdot^j\right), \tag{74}$$

where $\theta(f)$ is the solution of system (31) with the initial condition $\theta(f)|_Z = p^*(f)$. Denote by $\{\cdot, \cdot\}_{T^*M}$ the standard Poisson bracket on T^*M,

$$\{F, G\}_{T^*M} = \frac{\partial F}{\partial \xi_k}\frac{\partial G}{\partial x^k} - \frac{\partial G}{\partial \xi_k}\frac{\partial F}{\partial x^k}.$$

It turns out that S is a global Poisson morphism and T is a global anti-Poisson morphism from $(C^\infty(M), \{\cdot, \cdot\})$ to $(C^\infty(T^*M, Z), \{\cdot, \cdot\}_{T^*M})$, and for $f, g \in C^\infty(M)$ the elements Sf and Tg Poisson commute with respect to the Poisson bracket $\{\cdot, \cdot\}_{T^*M}$. Thus, according to [10], there exists a formal symplectic groupoid on (T^*M, Z) whose source mapping is S and target mapping is T. We call it *Fedosov's formal symplectic groupoid*.

One can prove the following lemma by straightforward calculations.

Lemma 8. *The space $\mathcal{H}_\Omega$ is closed with respect to the Poisson bracket $\{\cdot, \cdot\}_{T^*M}$.*

Thus $\mathcal{H}_\Omega$ has two different Poisson algebra structures corresponding to $\{\cdot, \cdot\}_\Omega$ and $\{\cdot, \cdot\}_{T^*M}$. We will prove that S and T are global mappings and that the images of S and T belong to $\mathcal{H}_\Omega$. The latter is a specific property of Fedosov's formal symplectic groupoids. First we need to prove a technical statement.

Lemma 9. *There exists an invertible formal matrix $A_k^i(x, \zeta)$ such that*

$$\pi^{ik} \frac{\partial \xi_k}{\partial \zeta_l} = A_k^i \pi^{kl}.$$

Proof. Using formulas (42), (71) and (73), we proceed as follows:

$$
\begin{aligned}
\pi^{ik} \frac{\partial \xi_k}{\partial \zeta_l} &= -\pi^{ik} P_t^l \frac{\partial u_k}{\partial \xi_t}(x, -\zeta_j P_.^j) - \pi^{ik} Q_t^l \frac{\partial u_k}{\partial \xi_t}(x, \zeta_j Q_.^j) \\
&= -\pi^{ts} P_t^l v_s^i(x, -\zeta_j P_.^j) - \pi^{ts} Q_t^l v_s^i(x, \zeta_j Q_.^j) \\
&= \left(Q_k^s v_s^i(x, -\zeta_j P_.^j) + P_k^s v_s^i(x, \zeta_j Q_.^j) \right) \pi^{kl}.
\end{aligned}
$$

We see from (53) and (71) that the matrix

$$A_k^i = Q_k^s v_s^i(x, -\zeta_j P_.^j) + P_k^s v_s^i(x, \zeta_j Q_.^j)$$

is invertible.   □

Denote by B_k^i the inverse of the matrix A_k^i. Lemma 9 implies the following formula:

$$B_j^i \pi^{jk} = \pi^{il} \frac{\partial \zeta_l}{\partial \xi_k}. \tag{75}$$

Lemma 10. *For a given local function f on M there exists a formal function $a_l(x, \xi)$ such that*

$$\frac{\partial(Sf)}{\partial \xi_k} = \pi^{kl} a_l.$$

Proof. It follows from Proposition 4 that

$$\frac{\partial \theta(f)}{\partial \xi_s} = \pi^{st} b_t \tag{76}$$

for some formal function $b_t(x, \xi)$. Differentiating both sides of the first formula in (74) with respect to ζ_l we get, using (76), the following equality:

$$\frac{\partial(Sf)}{\partial \xi_k} \frac{\partial \xi_k}{\partial \zeta_l} = -P_s^l \pi^{st} b_t \left(x, -\zeta_j P_.^j \right). \tag{77}$$

Taking into account (71), (75) and (76) we get from (77) that

$$\frac{\partial(Sf)}{\partial \xi_k} = -\pi^{kl} B_l^s Q_s^t b_t \left(x, -\zeta_j P_.^j \right)$$

whence the lemma follows.   □

Lemma 11. *Assume that $F(x, \xi)$ is a formal function such that*

$$\frac{\partial F}{\partial \xi_k} = \pi^{kl} a_l$$

and $\phi_p(x, \xi), \psi_p(x, \xi)$ are formal functions such that $\pi^{ip}\phi_p = \pi^{ip}\psi_p$. Then $F(x, \phi) = F(x, \psi)$.

Proof. Consider the function

$$\alpha(t) = F(x, t\phi + (1-t)\psi),$$

so that $\alpha(0) = F(x, \psi)$ and $\alpha(1) = F(x, \phi)$. We have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\partial F}{\partial \xi_k}(\phi_k - \psi_k) = \pi^{kl} a_l(\phi_k - \psi_k) = 0,$$

whence the lemma follows. $\quad\square$

The local potential u_p of the global vertical 1-form U is not uniquely defined. Assume that $\tilde{u}_p$ is another local potential of U. Then $\pi^{ip} u_p = \pi^{ip} \tilde{u}_p$. It follows from Lemmas 10 and 11 that formulas (73) and (74) determine the same mapping S for the potentials u_p and $\tilde{u}_p$. Thus S is globally defined. Lemma 10 implies that the image of S is in the space $\mathcal{H}_\Omega$. The proof that T is globally defined and has the image in $\mathcal{H}_\Omega$ is similar.

The following theorem can be proved by long straightforward calculations.

Theorem 4. *The mappings*

$$S, T : (C^\infty(M), \{\cdot, \cdot\}) \to (\mathcal{H}_\Omega, \{\cdot, \cdot\}_{T^*M})$$

are a Poisson and an anti-Poisson morphism, respectively. Moreover, the images of the mappings S and T Poisson commute. Thus there is a unique formal symplectic groupoid on (T^*M, Z) whose source and target maps are S and T, respectively.

7. The symplectic case

In [9] we gave a self-contained construction of the formal symplectic groupoid of Fedosov's deformation quantization. In this section we will show that in the symplectic case Fedosov's formal symplectic groupoid constructed in the previous sections is the same groupoid as in [9]. Recall the construction from [9].

Let M be a symplectic manifold with symplectic form $\omega = \frac{1}{2}\omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$. The inverse of ω_{ij} is a Poisson tensor π^{ij}. We assume that there is a tensor Λ^{ij} on M such that

$$\frac{1}{2}\left(\Lambda^{ij} - \Lambda^{ji}\right) = \pi^{ij}.$$

Let $\nabla_\bullet$ be a covariant connection which respects Λ^{ij} (and therefore the symplectic and Poisson tensors as well). The connection $\nabla_\bullet$ may have torsion. From these data one can construct Fedosov's deformation quantization on M and the corresponding formal symplectic groupoid (see [9,10]). The groupoid construction starts with the (ν-free) Fedosov's lift

$$\tau^\vee : C^\infty(M) \to C^\infty(TM, Z)$$

first extracted from Fedosov's quantization in [4].

Introduce Fedosov's operators on $C^\infty(TM) \otimes \Omega(M)$, where $\Omega(M)$ is the space of differential forms on M:

$$\delta(a) = \mathrm{d}x^i \wedge \frac{\partial a}{\partial y^i}, \qquad \delta'(a) = y^i \, \iota\left(\frac{\partial}{\partial x^i}\right) a.$$

Here $\{x^i\}$ are local coordinates on M and $\{y^i\}$ are the corresponding fibre coordinates on TM. Denote by $\{\cdot, \cdot\}_{TM}$ the fibrewise Poisson bracket on TM,

$$\{F, G\}_{TM} = \pi^{jk} \frac{\partial F}{\partial y^j} \frac{\partial G}{\partial y^k}.$$

Connection $\nabla_\bullet$ on M induces a covariant connection $\bar\nabla_\bullet$ on (TM, Z) expressed locally as follows:

$$\bar\nabla_{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} - \Gamma_{ij}^k y^j \frac{\partial}{\partial y^k}.$$

There exists a flat nonlinear covariant connection $D_\bullet$ on (TM, Z),

$$D_\bullet = \bar\nabla_\bullet + \{\rho, \cdot\}_{TM},$$

where $\rho = \rho_p(x, y)\mathrm{d}x^p$ is a unique 1-form satisfying the equation

$$-\omega = R + \bar\nabla_\bullet \rho + \frac{1}{2}\{\rho, \rho\}_{TM}$$

and such that $\delta'\rho = 0$ and $\rho_p = y^k \omega_{kp} \pmod{y^2}$. Here

$$R = \frac{1}{4}\omega_{s\alpha} R_{tkl}^\alpha y^s y^t \mathrm{d}x^k \wedge \mathrm{d}x^l,$$

where

$$R_{tkl}^s = \frac{\partial \Gamma_{lt}^s}{\partial x^k} - \frac{\partial \Gamma_{kt}^s}{\partial x^l} + \Gamma_{k\alpha}^s \Gamma_{lt}^\alpha - \Gamma_{l\alpha}^s \Gamma_{kt}^\alpha$$

is the curvature of $\nabla_\bullet$. In the notations of [9],

$$\rho_p = y^k \omega_{kp} + r_p^\vee.$$

Fedosov's lift $F = \tau^\vee(f)$ of a function $f \in C^\infty(M)$ is a unique element of $C^\infty(TM, Z)$ such that $D_\bullet F = 0$ and $F|_{y=0} = f$. The mapping $\tau^\vee$ is a Poisson morphism from $C^\infty(M)$ to $(C^\infty(TM, Z), \{\cdot, \cdot\}_{TM})$. Consider the global diffeomorphism of (T^*M, Z), $(x, \zeta) \mapsto (x, \xi)$, such that

$$\xi_p = \rho_p\left(x, \frac{1}{2}\Lambda^{j\cdot}\zeta_j\right) - \rho_p\left(x, \frac{1}{2}\Lambda^{j\cdot}\zeta_j\right).$$

Then, according to [9], the source and the target mappings of the formal symplectic groupoid of Fedosov's deformation quantization are given by the formulas

$$(Sf)(x, \xi) = \tau^\vee(f)\left(x, \frac{1}{2}\Lambda^{j\cdot}\zeta_j\right) \qquad \text{and} \qquad (Tf)(x, \xi) = \tau^\vee(f)\left(x, \frac{1}{2}\Lambda^{j\cdot}\zeta_j\right),$$

where $f \in C^\infty(M)$. To obtain the construction of Fedosov's formal symplectic groupoid one has to pull back Fedosov's lift via the mapping #. For a function $F = F(x, y) \in C^\infty(TM, Z)$ denote its pullback via # by $F^\#$ so that

$$F^\#(x, \xi) = F(x, \xi_j \pi^{j\cdot}).$$

We get the following identifications. For $f \in C^\infty(M)$

$$\theta(f) = \left(\tau^\vee(f)\right)^\#, u_p = -\rho_p^\#, \qquad \text{and} \qquad v_p^q = -\left(\pi^{qs}\frac{\partial \rho_s}{\partial y^p}\right)^\#.$$

Finally,

$$P_k^j = \frac{1}{2}\omega_{ki}\Lambda^{ij} \qquad \text{and} \qquad Q_k^j = \frac{1}{2}\Lambda^{ji}\omega_{ik}.$$

Notice that in the symplectic case $u_p dx^p$ is a global object on M, while in the general Poisson case only $\pi^{pq}u_q\frac{\partial}{\partial x^p}$ is global.

8. The Kähler–Poisson case

Assume that M is a Kähler–Poisson manifold with the Kähler–Poisson tensor $g^{\bar{l}k}$, where $k, \bar{l}$ are holomorphic and antiholomorphic indices, respectively (we use the agreement that $\pi^{\bar{l}k} = g^{\bar{l}k}$, $\pi^{k\bar{l}} = -g^{\bar{l}k}$, $\pi^{km} = 0$ and $\pi^{\bar{l}\bar{n}} = 0$). Denote by $\nabla^\bullet$ the Kähler–Poisson contravariant connection whose Christoffel symbols are given by (10). The Kähler–Poisson contravariant connection is Poisson and torsion-free. Introduce tensors P and Q on M by formula (72). With these data, one can construct a Fedosov's formal symplectic groupoid on (T^*M, Z). It was proved in [10] that for a Kähler–Poisson manifold M there exists a unique formal symplectic groupoid with separation of variables on (T^*M, Z), i.e., such that for a local holomorphic function a and a local antiholomorphic function b on M

$$Sa = a \qquad \text{and} \qquad Tb = b. \tag{78}$$

We want to prove the following proposition.

Proposition 5. *The Fedosov's formal symplectic groupoid on (T^*M, Z) constructed from the data $(\nabla^\bullet, P, Q)$ is the formal symplectic groupoid with separation of variables.*

For local holomorphic coordinates $z^k, \bar{z}^l$ on M denote by $\eta_k, \bar{\eta}_l$ the corresponding fibre coordinates on T^*M. Denote by $\mathcal{I}$ and $\bar{\mathcal{I}}$ the ideals in $C^\infty(T^*M, Z)$ locally generated by the coordinates η and $\bar{\eta}$, respectively. Formulas

$$\bar{\nabla}^{dz^k} = -g^{\bar{l}k}\frac{\partial}{\partial \bar{z}^l} + \frac{\partial g^{\bar{n}k}}{\partial \bar{z}^l}\bar{\eta}_n\frac{\partial}{\partial \bar{\eta}_l}, \qquad \bar{\nabla}^{d\bar{z}^l} = g^{\bar{l}k}\frac{\partial}{\partial z^k} - \frac{\partial g^{\bar{l}m}}{\partial z^k}\eta_m\frac{\partial}{\partial \eta_k}$$

imply that the connection $\bar{\nabla}^\bullet$ respects the ideals $\mathcal{I}$ and $\bar{\mathcal{I}}$. Let us show that the spaces $\mathcal{H}_\Omega \cap \mathcal{I}$ and $\mathcal{H}_\Omega \cap \bar{\mathcal{I}}$ are closed with respect to the Poisson bracket $\{\cdot, \cdot\}_\Omega$. Assume that $F, G \in \mathcal{H}_\Omega \cap \mathcal{I}$. Then there exist local potentials $a_k, a_{\bar{l}}$ and $b_k, b_{\bar{l}}$ of F and G, respectively, such that

$$\frac{\partial F}{\partial \bar{\eta}_l} = g^{\bar{l}k}a_k, \qquad \frac{\partial F}{\partial \eta_k} = -g^{\bar{l}k}a_{\bar{l}}, \qquad \frac{\partial G}{\partial \bar{\eta}_l} = g^{\bar{l}k}b_k, \qquad \frac{\partial G}{\partial \eta_k} = -g^{\bar{l}k}b_{\bar{l}}.$$

Now,

$$\{F, G\}_\Omega = g^{\bar{l}k}a_{\bar{l}}b_k - g^{\bar{l}k}a_k b_{\bar{l}} = \frac{\partial G}{\partial \bar{\eta}_l}a_{\bar{l}} - \frac{\partial F}{\partial \bar{\eta}_l}b_{\bar{l}} \in \mathcal{I}.$$

The proof that $\mathcal{H}_\Omega \cap \bar{\mathcal{I}}$ is closed with respect to the Poisson bracket $\{\cdot, \cdot\}_\Omega$ is similar. Using these statements and recurrence formulas (66) and (70) one can prove by induction the following lemma.

Lemma 12. (a) *Let U be the normalized solution of the Fundamental Equation corresponding to the Kähler–Poisson connection $\bar{\nabla}^\bullet$ on M. Then its local potential u_k, $u_{\bar{l}}$ satisfies the condition*

$$u_k = -\eta_k, u_{\bar{l}} = -\bar{\eta}_l \,(\mathrm{mod}\ \mathcal{I} \cap \bar{\mathcal{I}}).$$

(b) *Let a and b be a local holomorphic and a local antiholomorphic function on M, respectively. Then*

$$\theta(a) - a \in \bar{\mathcal{I}} \qquad and \qquad \theta(b) - b \in \mathcal{I}.$$

Lemma 12 implies that for the tensors P and Q given by formula (72) the change of variables (73) is trivial. Namely, for $u_k = u_k(z, \bar{z}, \eta, \bar{\eta})$ and $u_{\bar{l}} = u_{\bar{l}}(z, \bar{z}, \eta, \bar{\eta})$

$$u_k(z, \bar{z}, -\eta, 0) - u_k(z, \bar{z}, 0, \bar{\eta}) = \eta_k \qquad \text{and} \tag{79}$$
$$u_{\bar{l}}(z, \bar{z}, -\eta, 0) - u_{\bar{l}}(z, \bar{z}, 0, \bar{\eta}) = \bar{\eta}_l.$$

Now conditions (78) follow from formulas (74) and (79), and Lemma 12, which proves Proposition 5.

Acknowledgments

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